Time or Expence to make, and therefore might be supplied by new ones, as often as they happen'd to break.

These Moulds seem to have been burnt or baked sufficiently to make them hard, but not so as to render them porous like Bricks, whereby they would have lost their smooth and even Surface; which in these is plainly so close, that whatever Metal should be formed in them would have no Appearances like the Sand-Holes, by which counterfeit Coins or Medals are usually detected.

London, May 20.
1747.

XXVI. *An extract of a letter from* William Jones *Esq; F. R. S. to* Martin Folkes *Esq; President of the* Royal Society; *containing a commodious disposition of equations for exhibiting the relations of goniometrical lines.*

THEOREM.

Presented July 4. 1747.

In a circle whose radius is r, *let there be two arcs,* A *the greater,* a *the less, each in the first quadrant; put* s, t, ſ, *and* v, *for the sine, tangent, secant, and versed sine of an arc;* s', t', ſ', *the sine, tangent, secant of the complement, and* v', *the versed sine of the supplement of that arc; let* $z=\frac{1}{2}\overline{A+a}$, $x=\frac{1}{2}\overline{A-a}$; *or if* z *and* x *be put for the arcs, it will be* $A=z+x$, $a=z-x$.

Then will the terms in any column of the following table, be proportional to their corresponding ones in any other column.

From

A TABLE of the Relations of Goniometrical Lines.

$2s,z$	$\frac{s,2z}{s,\overline{A+a}}$	$\frac{v,2z}{v,\overline{A+a}}$	$s,A+s,a$	$\frac{\grave{s},a-\grave{s},A}{v,A-v,a}$
$2s,x$	$s,A-s,a$	$\frac{\grave{s},a-\grave{s},A}{v,A-v,a}$	$\frac{s,2x}{s,\overline{A-a}}$	$\frac{v,2x}{v,\overline{A-a}}$
$2\grave{s},z$	$\frac{\grave{v},2z}{\grave{v},\overline{A+a}}$	$\frac{s,2z}{s,\overline{A+a}}$	$\grave{s},A+\grave{s},a$	$s,A-s,a$
$2\grave{s},x$	$\grave{s},A+\grave{s},a$	$s,A+s,a$	$\frac{\grave{v},2x}{\grave{v},\overline{A-a}}$	$\frac{s,2x}{s,\overline{A-a}}$
r	$\grave{s},z$	s,z	$\grave{s},x$	s,x
$\text{ſ},z$	r	t,z		
t,z	s,z	$\text{ſ},z-\grave{s},z$		
$\grave{\text{ſ}},z$	$\grave{t},z$	r		
$t,z+\grave{t},z$	$\grave{\text{ſ}},z$	$\text{ſ},z$		
$\text{ſ},x$			r	t,x
t,x			s,x	$\text{ſ},x-\grave{s},x$

From hence, almoſt an infinite number of theorems may eaſily be derived; ſome of which are the following, given here as examples of the uſe of the table.

I. $s,z\times s,x=\tfrac{1}{2}r\times\overline{\grave{s},a-\grave{s},A}=\tfrac{1}{2}r\times\overline{\grave{s},\overline{z-x}-\grave{s},\overline{z+x}}=\frac{s,z}{\text{ſ},x}rr=\frac{s,x}{\text{ſ},z}rr.$

$\grave{s},z\times\grave{s},x=\tfrac{1}{2}r\times\overline{\grave{s},a+\grave{s},A}=\tfrac{1}{2}r\times\overline{\grave{s},\overline{z-x}+\grave{s},\overline{z+x}}=\frac{\grave{s},z}{\text{ſ},x}rr=\frac{\grave{s},x}{\text{ſ},z}rr.$

II. If A, B, C, be any three angles; $Z=A+B, X=A-B, H=\tfrac{1}{2}\overline{A+B+C}$.

Then $\tfrac{1}{2}r\times\overline{v,C-v,X}=s,\tfrac{1}{2}\overline{C+X}\times s,\tfrac{1}{2}\overline{C-X}=s,\tfrac{1}{2}\overline{A+C-B}\times s,\tfrac{1}{2}\overline{B+C-A}=s,\overline{H-B}\times s,\overline{H-A}.$

And $\tfrac{1}{2}r\times\overline{v,Z-v,C}=s,\tfrac{1}{2}\overline{Z+C}\times s,\tfrac{1}{2}\overline{Z-C}=s,\tfrac{1}{2}\overline{A+B+C}\times s,\tfrac{1}{2}\overline{A+B-C}=s,H\times s,\overline{H-C}.$

III. $\frac{ss,z}{\grave{s}\grave{s},z}=\frac{tt,z}{rr}=\frac{rr}{\grave{t}\grave{t},z}=\frac{v,2z}{\grave{v},2z}=\frac{t,z}{\grave{t},z}$; Or $\frac{ss,\frac{1}{2}z}{\grave{s}\grave{s},\frac{1}{2}z}=\frac{tt,\frac{1}{2}z}{rr}=\frac{rr}{\grave{t}\grave{t},\frac{1}{2}z}=\frac{v,z}{\grave{v},z}=\frac{t,\frac{1}{2}z}{\grave{t},\frac{1}{2}z}.$

IV.

IV. $\frac{1}{2}r = \frac{ss,z}{v,2z} = \frac{ss,\frac{1}{2}z}{v,z} = \frac{s`s`,z}{v`,2z} = \frac{s`s`,\frac{1}{2}z}{v`,z}$; and $s,z = \frac{2ss,\frac{1}{2}z}{t,\frac{1}{2}z} = \frac{2s`s`,\frac{1}{2}z}{t`,\frac{1}{2}z}$.

V. $\frac{s,z}{v,z} = \frac{r}{t,\frac{1}{2}z} = \frac{t`,\frac{1}{2}z}{r} = \frac{v`,z}{s,z}$.

VI. $\frac{t,z}{t,x} = \frac{s,A+s,a}{s,A-s,a} = \frac{t`,x}{t`,z}$; and $\frac{rr}{t,z \times t,x} = \frac{t`,z}{t,x} = \frac{t`,x}{t,z} = \frac{s`,a+s`,A}{s`,a-s`,A} = \frac{t`,z \times t`,x}{rr}$.

VII. $\frac{s,A}{s,a} = \frac{t,z+t,x}{t,z-t,x} = \frac{s,\overline{z+x}}{s,\overline{z-x}}$; if z and x are two arcs, then $A = z+x$, $a = z-x$.

VIII. $s,\overline{z \pm x} = \frac{s,z \times s`,x \pm s`,z \times s,x}{r} = \frac{t,z \pm t,x}{\int,z \times \int,x}$.

IX. $s`,\overline{z \pm x} = \frac{s`,z \times s`,x \mp s,z \times s,x}{r} = \frac{rr \mp t,z \times t,x}{\int,z \times \int,x} r$.

X. $t,\overline{z \pm x} = \frac{t,z \pm t,x}{rr \mp t,z \times t,x} rr$; and $t`,\overline{z \pm x} = \frac{rr \mp t,z \times t,x}{t,z \pm t,x}$.

XI. $\int,\overline{z \pm x} = \frac{\int,z \times \int,x}{rr \mp t,z \times t,x} r$; and $\int`,\overline{z \pm z} = \frac{\int,z \times \int,x}{t,z \pm t,x}$.

XII. In three equidifferent arcs A, z, a; where z ($= \frac{1}{2}\overline{A+a}$) is the mean arc, and x ($= \frac{1}{2}\overline{A-a}$) their common difference; put $p = \frac{s`,x}{r}$, $q = \frac{s,x}{r}$; $P = 2p \times s,z$, $Q = 2q \times s`,z$.

Then $s,A = P - s,a = Q + s,a$; And $s,a = P - s,A = s,A - Q$.

XIII. Let $d = v,A - v,a = s`,a - s`,A$; then $ss,A - ss,a = 2s`,\overline{A+d} \times d = 2s`,\overline{a-d} \times d$.

Note, When an arc is terminated in the second, third, or fourth quadrant, some of the signs (+ and −) of the terms in the preceding theorems, will, by the known rules, become contrary to what they now are.

XIV.

XIV. Let A, B, C, &c. be the sines, a, b, c, &c. the co-sines, a', b', c', &c. the tangents, of the arcs, α, β, γ, &c. whose number is n; the radius being r; put S for the product of the n co-sines, S', S'', S''', &c. for the sum of the products made of every sine, every two, three, &c. sines, by the other ($n-1, n-2, n-3$, &c.) co-sines, where the co-sine noted by $n-n$ is unity.

Then the sine of $\overline{\alpha+\beta+\gamma+\delta, \&c.} = \overline{S'-S'''+S^{v}-S^{vii}, \&c.} \times \frac{1}{r^{n-1}}$.

And the co-sine of $\overline{\alpha+\beta+\gamma+\delta, \&c.} = \overline{S-S''+S^{iv}-S^{v}, \&c.} \times \frac{1}{r^{n-1}}$.

XV. Also putting T' for the sum of the tangents of the arcs, α, β, γ, &c. T'', T''', T^{iv}, &c. for the sum of the products of every two, three, four, &c. tangents; and

$$A = T'$$
$$B = AT'' - T'''$$
$$C = BT'' - AT^{iv} + T^{v}$$
$$D = CT'' - BT^{iv} + AT^{vi} - T^{vii}$$
$$E = DT'' - CT^{iv} + BT^{vi} - AT^{viii} + T^{ix}. \quad \text{Put } R = \frac{1}{rr}$$
&c.

Then the tangent of $\overline{\alpha+\beta+\gamma+\delta, \&c.} = A + BR + CR^2 + DR^3 + ER^4$; &c.

XVI. Hence, the sine, tangent, and secant, of any arc a, being represented by s, t, f, the co-sine, co-tangent, and co-secant, by s', t', f'; those of the arc na are expressed as in the following theorems.

Putting $n' = n.\frac{n-1}{2}$; $n'' = n'.\frac{n-2}{3}$; $n''' = n''.\frac{n-3}{4}$; $n^{v} = n'''.\frac{n-4}{5}$; &c.

Sine of $na = \overline{nA - n''AP + n^{v}BP - n^{vi}CP + n^{viii}DP, \&c.} \times \frac{s'^{n-1}}{r^{n-1}}$;

where $P = \frac{ss}{s's'}$; $A = s$; $B = AP$; $C = BP$; $D = CP$; &c.

Or $= \overline{ns - \frac{n-1}{2}.\frac{n-2}{3}AP + \frac{n-3}{4}.\frac{n-4}{5}BP - \frac{n-5}{6}.\frac{n-6}{7}CP \ \&c.} \times \frac{s'^{n-1}}{r^{n-1}}$;

where A, B, C, &c. stand for the respective preceding terms.

Or $= ns + \frac{1+n}{2}.\frac{1-n}{3}AQ + \frac{3+n}{4}.\frac{3-n}{5}BQ + \frac{5+n}{6}.\frac{5-n}{7}CQ + \frac{7+n}{8}.\frac{7-n}{9}DQ$. &c.

where $Q = \frac{ss}{rr}$; A, B, C, &c. stand as before.

XVII. Co-fine of $na = \overline{1 - nP + n^{\prime\prime\prime}P^2 - n^{v}P^3 + n^{vii}P^4, \&c.} \times \frac{s^n}{r^{n-1}}$, where $P = \frac{ss}{tt}$

Or $= r + \frac{0+n}{1} \cdot \frac{0-n}{2} AQ + \frac{2+n}{3} \cdot \frac{2-n}{4} BQ + \frac{4+n}{5} \cdot \frac{4-n}{6} CQ + \frac{6+n}{7} \cdot \frac{6-n}{8} DQ, \&c.$

where $Q = \frac{ss}{r}$; and $A, B, C, \&c.$ ftand for the refpective preceding terms.

Or put $M = \overline{\frac{2s}{r}}\Big|^n \times r$; $N = \frac{rr}{4ss}$; $A = \frac{1}{2}$; $B = AN$; $C = BN$; $D = CN, \&c$; $p = n$; $p' = n - 1$; $p'' = n - 2, \&c.$

And $a' = p$; $b' = p \cdot \frac{1}{2} p^{\prime\prime\prime}$; $c' = p \cdot \frac{1}{2} p^{iv} \cdot \frac{1}{3} p^{v}$; $d' = p \cdot \frac{1}{2} p^{v} \cdot \frac{1}{3} p^{vi} \cdot \frac{1}{4} p^{vii}$; $e' = p \cdot \frac{1}{2} p^{vi} \cdot \frac{1}{3} p^{vii} \cdot \frac{1}{4} p^{viii} \cdot p \frac{1}{5} p^{ix}$; $\&c.$

The co-fine of $na = \overline{A - Ba' + Cb' - Dc' + Ed', \&c.} \times M.$

XVIII. Let $A = -n^{\prime\prime} + nn'$
$B = +n^{iv} - nn^{\prime\prime\prime} + An'$
$C = -n^{vi} + nn^{v} + Bn' - An^{\prime\prime\prime}$
$D = +n^{viii} - nn^{vii} + Cn' - Bn^{\prime\prime\prime} + An^{v}$
$\&c.$

$A' = \frac{1}{n} \cdot n^{\prime\prime} - n'$
$B' = \frac{1}{n} \cdot \overline{n^{\prime\prime} A' - n^{iv}} + n^{\prime\prime\prime}$
$C' = \frac{1}{n} \cdot \overline{n^{\prime\prime} B' - n^{iv} A' + n^{vi}} - n^{v}$
$D' = \frac{1}{n} \cdot \overline{n^{\prime\prime} C' - n^{iv} B' + n^{vi} A' - n^{viii}} + n^{vii}$
$\&c.$

The tangent of $na = nt + At^3 r^{-2} + Bt^5 r^{-4} + Ct^7 r^{-6} + Dt^9 r^{-8}\ \&c.$

Or $= \overline{n + AN + BN^2 + CN^3 + DN^4 + \&c.} \times t$, where $N = \frac{tt}{rr}$

Or $= na' + Ab' + Bc' + Cd' + De', \&c.$ where $a' = t$; $b' = Na'$; $c' = Nb'$; $d' = Nc'$; $\&c.$

Or $= \frac{n - n^{\prime\prime\prime}N + n^{v}N^2 - n^{vii}N^3 + n^{ix}N^4, \&c.}{1 - n^{\prime\prime}N + n^{iv}N^2 - n^{vi}N^3 + n^{viii}N^4, \&c.} \times t.$

Co-tang^t of $na = \overline{r^2 + A't^2 + B't^2 N + C't^2 N^2 + D't^2 N^3 + E't^2 N^4 \&c.} \times \frac{rr}{nt}$; where $N = \frac{tt}{rr}$

Or $= \overline{1 + A'N + B'N^2 + C'N^3 + D'N^4 + E'N^5, \&c.} \times \frac{1}{n} r^2 t$; where $N = \frac{rr}{tt}$

Or $= \frac{1 - n^{\prime\prime}N + n^{iv}N^2 - n^{vi}N^3 + n^{viii}N^4 - n^{x}N^5, \&c.}{n - n^{\prime\prime\prime}N + n^{v}N^2 - n^{vii}N^3 + n^{ix}N^4 - n^{xi}N^5, \&c.} \times \frac{rr}{t}$; where $N = \frac{tt}{rr}$

XIX. Let $A = n'$
$B = An' - n^{\prime\prime\prime}$
$C = Bn' - An^{\prime\prime\prime} + n^{v}$
$D = Cn' - Bn^{\prime\prime\prime} + An^{v} - n^{vii}$
$\&c.$

$A' = \frac{1}{n} \cdot n^{\prime\prime}$
$B' = \frac{1}{n} \cdot \overline{n^{\prime\prime} A' - n^{iv}}$
$C' = \frac{1}{n} \cdot \overline{n^{\prime\prime} B' - n^{iv} A' + n^{vi}}$
$D' = \frac{1}{n} \cdot \overline{n^{\prime\prime} C' - n^{iv} B' + n^{vi} A' - n^{viii}}$
$\&c.$

Secant of $na = \overline{1 + AN + BN^2 + CN^3 + DN^4 + EN^5, \&c.} \times M.$

Or $= \frac{1}{1 - n^{\prime\prime}N + n^{iv}N^2 - n^{vi}N^3 + n^{viii}N^4, \&c.} \times M$; where $N = \frac{tt}{rr}$, $M = \frac{rf^n}{rn}$

Co-fecant of $na = \overline{1 + A'N + B'N^2 + C'N^3 + D'N^4 + E'N^5, \&c.} \times M$; where $N = \frac{tt}{rr}$, $M = \frac{rrf^n}{ntr^n}$

Or $= \frac{1}{n - n^{\prime\prime\prime}N + n^{v}N^2 - n^{vii}N^3 + n^{ix}N^4, \&c.} \times M$; where $N = \frac{tt}{rr}$, $M = \frac{r^{n-2}}{tf^{n-2}}$

XX.

XX. Let c be the chord of an arc (a) of the circumference of a circle, whose diameter is d. Put $N = \frac{cc}{dd}$.

The chord of $na = nc + \frac{1+n}{2} \cdot \frac{1-n}{3} AN + \frac{3+n}{4} \cdot \frac{3-n}{5} BN + \frac{5+n}{6} \cdot \frac{5-n}{7} CN + \frac{7+n}{8} \cdot \frac{7-n}{9} DN$, &c. where A, B, C, &c. ſtand for the reſpective preceding terms.

As the preceding theorems are eaſily deduced from the firſt, ſo the following are moſt readily ſeen to be the immediate conſequences of theſe; and all depending upon no other principles than what are generally made uſe of in common computations.

XXI. Putting s, s', t, t', $\int$, $\int'$, for the ſine, co-ſine, tangent, co-tangent, ſecant, co-ſecant, of an arc (a), and v its verſed ſine; let $q' = \frac{1}{2}$; $q'' = \frac{1}{3}q'$; $q''' = \frac{1}{4}q''$; $q^{iv} = \frac{1}{5}q'''$; $q^{v} = \frac{1}{6}q^{iv}$; &c. $N = \frac{aa}{rr}$.

Then $s = \overline{1 - q''N + q^{iv}N^2 - q^{vi}N^3 + q^{viii}N^4 + q^{x}N^5, \text{ \&c.}} \times a$.

$= a - q''a^3r^{-2} + q^{iv}a^5r^{-4} - q^{vi}a^7r^{-6} + q^{viii}a^9r^{-8}$, &c.

$= a - \frac{1}{2.3} AN + \frac{1}{4.5} BN - \frac{1}{6.7} CN + \frac{1}{8.9} DN$, &c. where A, B, C, &c. ſtand for the reſpective preceding terms.

And $s' = r - q'a^2r^{-1} + q'''a^4r^{-3} - q^{v}a^6r^{-5} + q^{vii}a^8r^{-7}$, &c.

$= \overline{1 - q'N + q'''N^2 - q^{v}N^3 + q^{vii}N^4 \quad q^{x}N^5, \text{ \&c.}} \times r$.

$= r - \frac{1}{1.2} a^2r^{-1} + \frac{1}{3.4} AN - \frac{1}{5.6} BN + \frac{1}{7.8} CN$, &c. A, B, C, &c. as before.

XXII. Alſo $v = q'a^2r^{-1} - q^{iv}a^4r^{-3} + q^{vi}a^6r^{-5} - q^{viii}a^8r^{-7}$, &c.

$= \frac{1}{1.2} a^2r^{-1} - \frac{1}{3.4} AN - \frac{1}{5.6} BN - \frac{1}{7.8} CN - \frac{1}{9.10} DN$, &c.

$= \overline{\frac{1}{1.2}N - \frac{1}{3.4} AN - \frac{1}{5.6} BN - \frac{1}{7.8} CN, \text{ \&c.}} \times r$. A, B, C, &c. as before.

XXIII. Let $A = +q' - q''$
$B = -q''' + q^{iv} + Aq'$
$C = +q^{v} - q^{vi} + Bq' - Aq'''$
$D = -q^{vii} + q^{viii} + Cq' - Bq''' + Aq^{v}$.
&c.

And $A' = -A$
$B' = -B - AA'$
$C' = -C - BA' - AB'$
$D' = -D - CA' - BB' - AC'$.
&c.

Tangent $t = a + Aa^3r^{-2} + Ba^5r^{-4} + Ca^7r^{-6} + Da^9r^{-8}$, &c.

Or $= \overline{1 + AN + BN^2 + CN^3 + DN^4 + EN^5, \text{ \&c.}} \times a$.

Co-tangent $t' = a^{-1}r^2 + A'a + B'a^3r^{-2} + C'a^5r^{-4} + D'a^7r^{-6}$, &c.

Or $= \overline{rr + A'a^2 + B'Na^2 + C'N^2a^2 + D'N^3a^2, \text{ \&c.}} \times \frac{1}{a}$.

XXIV.

XXIV. Alfo let $\alpha = +q'$
$\beta = -q''' + \alpha q'$
$\gamma = +q^{v} - \alpha q''' + \beta q'$
$\delta = -q^{vii} + \alpha q^{v} - \beta q''' + \gamma q'$.
&c.

And $\alpha' = +q''$
$\beta' = -q^{iv} + \alpha' q''$
$\gamma' = +q^{vi} - \alpha' q^{iv} + \beta' q''$
$\delta' = -q^{viii} + \alpha' q^{vi} - \beta' q^{iv} + \gamma' q''$.
&c.

Secant $s = r + \alpha a^2 r^{-1} + \beta a^4 r^{-3} + \gamma a^6 r^{-5} + \delta a^8 r^{-7}$, &c.

Or $= \overline{1 + \alpha N + \beta N^2 + \gamma N^3 + \delta N^4, \&c.} \times r$.

Co-fecant $\grave{s} = a^{-1} r^2 + \alpha' a + \beta' a^3 r^{-2} + \gamma' a^5 r^{-4} + \delta' a^7 r^{-6}$, &c.

Or $= \overline{rr + \alpha' aa + \beta' Naa + \gamma' N^2 aa + \delta' N^3 aa, \&c.} \times \frac{1}{a}$. where $N = \frac{aa}{rr}$;

XXV. Putting $p' = \frac{1}{2}$; $p'' = \frac{3}{4}p$; $p''' = \frac{5}{6}p''$; $p^{iv} = \frac{7}{8}p'''$; $p^{v} = \frac{9}{10}p^{iv}$; &c. $N = \frac{ss}{rr}$

Then arc $a = \overline{1 + \frac{1}{3}p'N + \frac{1}{5}p''N^2 + \frac{1}{7}p'''N^3 + \frac{1}{9}p^{iv}N^4, \&c.} \times s$.

Or $= s + \frac{1}{3}p'AN + \frac{1}{5}p''BN + \frac{1}{7}p'''CN + \frac{1}{9}p^{iv}DN$, &c.

Or $= s + \frac{1.1}{2.3}AN + \frac{3.3}{4.5}BN + \frac{5.5}{6.7}CN + \frac{7.7}{8.9}DN$, &c. where A, B, C, &c. ftand for the refpective preceding terms.

XXVI. If v is the verfed fine of an arc a, diameter being d, $M = \frac{v}{d}$, $R = \sqrt{dv}$.

Then arc $a = \overline{1 + \frac{1.1}{2.3}M + \frac{3.3}{4.5}AM + \frac{5.5}{6.7}BM + \frac{7.7}{8.9}CM, \&c.} \times R$; A, B, C, &c. are as before.

XXVII. And putting $N = \frac{tt}{rr}$, $A = t$, $B = AN$, $C = BN$, $D = CN$, &c.

Then arc $a = t - \frac{1}{3}AN + \frac{1}{5}BN - \frac{1}{7}CN + \frac{1}{9}DN + \frac{1}{11}EN$, &c.

Or $= \overline{1 - \frac{1}{3}N + \frac{1}{5}N^2 - \frac{1}{7}N^3 + \frac{1}{9}N^4 + \frac{1}{11}N^5, \&c.} \times t$.

XXVIII. Alfo, if c is the chord of an arc (a); and $N = \frac{cc}{dd}$.

Then arc $a = c + \frac{1.1}{2.3}AN + \frac{3.3}{4.5}BN + \frac{5.5}{6.7}CN + \frac{7.7}{8.9}DN$, &c. where A, B, C, &c. ftand for the refpective preceding terms.